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Numerical inversion of a characteristic function

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SUMMARY

A method is described for finding a bound on the error when a version of the usual characteristic function inversion formula is evaluated by numerical integration. The method is applied to the calculation of the distribution function of a quadratic form in normal random variables.

Some key words: Numerical inversion of characteristic function; Quadratic form in normal variables; Trapezoidal rule.

1. Introduction

Suppose that a real random variable, X, has a readily calculable characteristic function

$$\phi(u) = E(e^{iuX}), \tag{1}$$

and that we wish to compute the probability $\mathrm{pr}\,(X < x)$.

In this note a method is presented for finding a bound on the integration error when Gil-Pelaez's (1951) inversion formula is evaluated by numerical integration using the trapezoidal rule. This method is very similar to those suggested by Bohmann (1961, 1970, 1972) but is more suitable for problems such as the one considered in § 4.

2. Continuous case

In this section we will suppose that, for some c and $\delta > 0$ and for all $u > 1$,

$$|\phi(u)| < cu^{-\delta} \tag{2}$$

and that $E(|X|) < \infty$. It follows (Gil-Pelaez, 1951) that

$$\mathrm{pr}\,(X < x) = \tfrac{1}{2} - \int_{-\infty}^{\infty} \mathrm{Im}\left(\frac{\phi(u)\, e^{-iux}}{2\pi u}\right) du \tag{3}$$

and hence that

$$\mathrm{pr}\,(X < x-t) - \mathrm{pr}\,(X > x+t) = -2\int_{-\infty}^{\infty} \mathrm{Im}\left(\frac{\phi(u)\, e^{-iux}}{2\pi u}\right) \cos(ut)\, du. \tag{4}$$

Equations (3) and (4) may be rewritten as

$$\mathrm{pr}\,(X < x) - \tfrac{1}{2} = \int_0^{\Delta} S(u)\, du,$$

$$\mathrm{pr}\,(X < x - 2\pi n/\Delta) - \mathrm{pr}\,(X > x + 2\pi n/\Delta) = 2\int_0^{\Delta} S(u) \cos(2\pi n u/\Delta)\, du,$$

where

$$S(u) = -\sum_{k=-\infty}^{\infty} \mathrm{Im}\left\{\frac{\phi(u+k\Delta)\, e^{-i(u+k\Delta)\,x}}{2\pi(u+k\Delta)}\right\} \tag{5}$$

and n is a positive integer. Applying the Fourier cosine series summation formula, we obtain

$$\mathrm{pr}(X < x) + \sum_{n=1}^{\infty} \{\mathrm{pr}(X < x - 2\pi n/\Delta) - \mathrm{pr}(X > x + 2\pi n/\Delta)\} \cos(2\pi n u/\Delta)$$
$$= \tfrac{1}{2} - \Delta \sum_{k=-\infty}^{\infty} \mathrm{Im}\left\{\frac{\phi(u+k\Delta)\, e^{-i(u+k\Delta)x}}{2\pi(u+k\Delta)}\right\}. \quad (6)$$

Useful formulae are obtained by setting u equal to either 0 or $\frac{1}{2}\Delta$. Substituting $u = \frac{1}{2}\Delta$, one obtains

$$\mathrm{pr}(X < x) + \sum_{n=1}^{\infty} (-1)^n \{\mathrm{pr}(X < x - 2\pi n/\Delta) - \mathrm{pr}(X > x + 2\pi n/\Delta)\}$$
$$= \tfrac{1}{2} - \sum_{k=0}^{\infty} \mathrm{Im}\,[\phi\{(k+\tfrac{1}{2})\Delta\} e^{-i(k+\frac{1}{2})\Delta x}]/\{\pi(k+\tfrac{1}{2})\}. \quad (7)$$

Thus to compute $\mathrm{pr}(X < x)$, one chooses Δ so that

$$\max\{\mathrm{pr}(X < x - 2\pi/\Delta), \mathrm{pr}(X > x + 2\pi/\Delta)\} \quad (8)$$

is less than, say, half the maximum allowable error and then computes

$$\tfrac{1}{2} - \sum_{k=0}^{K} \mathrm{Im}\,[\phi\{(k+\tfrac{1}{2})\Delta\} e^{-i(k+\frac{1}{2})\Delta x}]/\{\pi(k+\tfrac{1}{2})\}, \quad (9)$$

where K is chosen so that the truncation error is also less than half the maximum allowable error.

There are many ways of finding bounds on (8). For example, suppose that the moment generating function, $E\{\exp(uX)\}$, exists in a neighbourhood (U_1, U_2) of the origin and $\psi(u)$ denotes its logarithm. Then by Feller (1966, p. 525), or by considering the expectation of $I_{X>x} - \exp\{u(X-x)\}$ and letting $x = \psi'(u)$, we have

$$\mathrm{pr}\{X > \psi'(u)\} \leqslant \exp\{\psi(u) - u\psi'(u)\} \quad (U_2 > u > 0),$$

with a similar expression for the lower tail, I denoting an indicator function.

3. Discrete case

Suppose that X can take on only integer values and $E(|X|) < \infty$. If $\phi(u) = E(e^{iuX})$ and x is an integer, then

$$\mathrm{pr}(X < x) = \tfrac{1}{2} - \int_{-\pi}^{\pi} \mathrm{Re}\left\{\frac{\phi(u)\, e^{-iux}}{2\pi(1 - e^{-iu})}\right\} du$$

and the method proceeds as in the continuous case.

4. Distribution of a quadratic form in independent normal random variables

A quadratic form in independent normal random variables may be expressed as the linear combination

$$Q = \sum_{i=1}^{r} \lambda_i X_i + \sigma X_0, \quad (10)$$

where X_i has a chi-squared distribution with n_i degrees of freedom and noncentrality parameter $\delta_i^2 (i = 1, \ldots, r)$, X_0 has a standard normal distribution and all the X_i are independent. See Johnson & Kotz (1970, Ch. 29) for a discussion of other methods for finding the distribution of Q and for applications. The numerical integration method has been suggested by Imhoff (1961) but he did not give a method for selecting the sampling interval. The characteristic function

$$E(e^{iuQ}) = \frac{\exp\left\{iu \sum_{j=1}^{r} \lambda_j \delta_j^2 \Big/ (1-2iu\lambda_j) - \tfrac{1}{2}u^2\sigma^2\right\}}{\prod_{j=1}^{r} (1-2iu\lambda_j)^{\frac{1}{2}n_j}}$$

Table 1. *Number of terms to give accuracy of* 0·00005 *in numerical integration of chi-squared distribution*

Degrees of freedom n	pr $(\chi_n^2 < x)$ 0·001	0·5	0·999
1	> 20,000	7,238	917
2	16,670	685	281
3	1,044	209	120
4	224	92	72
5	101	53	51
6	51	51	31
10	19	16	13
100	8	6	7
1,000	7	5	7

may be readily calculated and so the method of § 2 may be followed. The calculation of the truncation point, K, although elementary, involves checking a number of different bounds and will not be discussed here. It is possible, in principle, for the terms of (9) to become large with fluctuating sign leading to a large roundoff error and this should be checked when (9) is evaluated. However, this possibility has not been found to be a problem in the numerous quadratic forms on which this method has been tried.

A computer program based on the ideas described here for calculating the distribution of (10) has been written in ALGOL and some idea of its practicality may be gained by using it to compute the distribution of a central chi-squared random variable for various numbers of degrees of freedom, n. The number of terms in the sum (9) used to give an accuracy of 0·00005 for various values of x, chosen so that $\text{pr}(\chi_n^2 < x) = 0{\cdot}001, 0{\cdot}5, 0{\cdot}999$, and n are given in Table 1. Nonzero values of δ_i and σ help and in general the method, in the simple form described here, is reasonable for about 4 decimal figures of accuracy if the sum (10) is not dominated by 1, 2 or 3 terms each with one degree of freedom, or by one term with three degrees of freedom, etc. It is also not an unreasonable way for calculating noncentral chi-squared and F probabilities.

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